doi:10.1006/jabr.2001.9026, available online at http://www.idealibrary.com on IDEAL®

Another Proof of Gluck's Theorem

Hiroshi Matsuyama

Department of Mathematics, Hyogo University of Teacher Education
E-mail: matsuyam@sci.hyogo-u.ac.jp

Communicated by George Glauberman

Received April 10, 2001

DEDICATED TO PROF. GEORGE GLAUBERMAN FOR HIS 60TH BIRTHDAY

Key Words: Gluck; finite; permutation; group; partition; regular.

In this article, G is a permutation group on a finite set Ω. We write permutations on the right, so that αg is the image of $\alpha \in \Omega$ by the action of $g \in G$. A subset S of Ω is said to be *G-regular* if the stabilizer $\{g \in G \mid Sg = S\}$ is the identity.

Our purpose is to give a direct short proof of the following theorem by Gluck [3, Corollary 3.3].

GLUCK'S THEOREM. *Let G be a permutation group of odd order on a finite set Ω. Then G has a regular subset in Ω.*

In [3, Theorem 1], Gluck proved that a primitive solvable permutation group has a regular set, except for 12 permutation groups. In his proof, Wolf's result on primitive solvable groups [4, Corollary 3.3] plays a role. By applying the odd order theorem [1, 2], Gluck's theorem above is an easy consequence of [3, Theorem 1].

In our proof, we do not need Wolf's result, but the odd order theorem is indispensable.

PROOF OF GLUCK'S THEOREM

Suppose to the contrary that there exists a permutation group G of odd order on a finite set Ω, such that G does not have a regular set. Among those counterexamples, we choose one such that $|\Omega|$ is minimal.

0021-8693/02 $35.00

CLAIM 1. *G is transitive.*

Proof. Suppose that G is not transitive. Let Ω_i $(i = 1, 2, \ldots, k)$ be G-orbits, and let $N_i = \{g \in G \mid \alpha g = \alpha \text{ for all } \alpha \in \Omega_i\}$. We note that $\bigcap_{i=1}^{k} N_i = 1$ and that $|\Omega_i| < |\Omega|$. If $|\Omega_i| > 1$, then there exists a G/N_i-regular subset S_i of Ω_i. Put $S_i = \Omega_i$ if $|\Omega_i| = 1$. Then the stabilizer of $S = \bigcup_{i=1}^{k} S_i$ is contained in $\bigcap_{i=1}^{k} N_i$. Hence S is G-regular, a contradiction. This proves the claim. ∎

CLAIM 2. *G is primitive.*

Proof. Suppose that G is not primitive. Then there exists blocks Δ_i $(i = 1, 2, \ldots, r)$, such that $1 < |\Delta_i| < |\Omega|$,

$$\Omega = \Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_r$$

is a disjoint union, and G permutes $\widetilde{\Omega} = \{\Delta_1, \Delta_2, \ldots, \Delta_r\}$ transitively. Let N be the kernel of the permutation representation of G on $\widetilde{\Omega}$. Since $|\widetilde{\Omega}| < |\Omega|$, G/N has a regular set $\widetilde{S}$. We may assume that $\widetilde{S} = \{\Delta_1, \Delta_2, \ldots, \Delta_s\}$. Let $S = \Delta_1 \cup \cdots \cup \Delta_s$. Then the stabilizer of S is N. Let N_i be the kernel of the permutation representation of N on Δ_i. We note that $\bigcap_{i=1}^{r} N_i = 1$. Since $1 < |\Delta_i| < |\Omega|$, there exists an N/N_i-regular subset T_i of Δ_i. Since $|\Delta_i|$ is odd, replacing T_i by $\Delta_i - T_i$ if necessary, we may assume that $|T_i| > |\Delta_i|/2$ for $i = 1, \ldots, s$, and $|T_j| < |\Delta_j|/2$ for $j = s+1, \ldots, r$. We put $T = T_1 \cup T_2 \cup \cdots \cup T_r$. Then the stabilizer M of T must stabilize S; hence we have that $M \subseteq N$. It follows that M stabilizes Δ_i and T_i, which implies that $M \subseteq N_i$. Thus we conclude that $M \subseteq \bigcap_{i=1}^{r} N_i$; it follows that $M = 1$. Therefore T is a G-regular set, which is a contradiction. This proves the claim. ∎

In the following, we let N be a minimal normal subgroup of G. Since G is solvable by the odd order theorem, N is an elementary abelian p-group for some odd prime p. Moreover, as is well known, N is a regular permutation group on Ω by virtue of Claim 2. Therefore, for α and β of Ω, there exists a unique element $x \in N$ such that $\alpha x = \beta$.

For $\alpha \in \Omega$ and $T \subseteq \Omega$, we define

$$T_\alpha = \{x \in N \mid \alpha x \in T\}.$$

Clearly T_α is a subset of N and $|T_\alpha| = |T|$.

CLAIM 3. $(T_\alpha)^g = (Tg)_{\alpha g}$ *for* $g \in G$.

Proof. If $x \in (Tg)_{\alpha g}$, then $\alpha g x \in Tg$. It follows that $gxg^{-1} \in T_\alpha$, which implies that $x \in (T_\alpha)^g$. Thus we have that $(T_\alpha)^g \supseteq (Tg)_{\alpha g}$. Since the converse inclusion is obtained in a similar way, the claim is proved. ∎

CLAIM 4. $T_{\alpha x} = x^{-1} T_\alpha$ *for* $x \in N$.

Proof. If $y \in T_{\alpha x}$, then $\alpha xy \in T$. Hence we have that $xy \in T_\alpha$, which implies that $y \in x^{-1}T_\alpha$. Thus we conclude that $T_{\alpha x} \subseteq x^{-1}T_\alpha$. In a similar way, we obtain that $T_{\alpha x} \supseteq x^{-1}T_\alpha$, which proves the claim. ∎

We note that $G_\alpha = \{g \in G \mid \alpha g = \alpha\}$ and that $G = G_\alpha N$.

CLAIM 5. *If $T_\alpha = (T_\alpha)^{-1}$ and $Tg = T$ for some $g \in G - G_\alpha$, then $|T|$ is a multiple of p.*

Proof. Put $g = xy$ for $x \in G_\alpha$ and $y \in N$. Since $g \notin G_\alpha$, we have that $y \neq 1$. From Claim 3, we obtain

$$T_{\alpha y} = T_{\alpha xy} = (Tg)_{\alpha g} = (T_\alpha)^g = (T_\alpha)^{xy}.$$

On the other hand, we obtain from Claim 4 that $T_{\alpha y} = y^{-1}T_\alpha$. Hence we have that $y^{-1}T_\alpha = (T_\alpha)^{xy}$. Consequently we have that $T_\alpha = (T_\alpha)^x y$. Then, applying the assumption that $T_\alpha = (T_\alpha)^{-1}$, we conclude that $T_\alpha = (T_\alpha)^x y^{-1}$. These imply that $T_\alpha = T_\alpha y^2$. Now the order of y^2 is an odd prime p; it follows that p divides $|T_\alpha| = |T|$. This proves the claim. ∎

To obtain a final contradiction, we choose an element $\alpha \in \Omega$ and fix it. Then G_α is a permutation group on $T = \Omega - \{\alpha\}$. And, for each element β of T, there exists a unique element x_β of $T_\alpha = N - \{1\}$ such that $\alpha x_\beta = \beta$ and $\beta y = \alpha(x_\beta)^y$ for $y \in G_\alpha$. Thus, as is well known, we can identify the action of $y \in G_\alpha$ on T with the conjugate action of y on T_α.

On the other hand, since N is of odd order, T_α is a disjoint union of the pairs $\{x, x^{-1}\}$ with $x \in T_\alpha$. We let

$$\Lambda = \{\{x, x^{-1}\} \mid x \in T_\alpha\}.$$

Then G_α acts on Λ faithfully, because a pair $\{x, x^{-1}\}$ is fixed by $g \in G_\alpha$ if and only if both x and x^{-1} are fixed by g. From $|\Lambda| < |\Omega|$, G_α has a regular subset Γ in Λ. We let

$$U = \bigcup_{\{x, x^{-1}\} \in \Gamma} \{x, x^{-1}\}.$$

Obviously, U is a subset of T_α and $U = U^{-1}$. Let S be a corresponding subset of T such that $S_\alpha = U$. We note that $S_\alpha = (S_\alpha)^{-1}$. By the assumption that G has no regular set, there exists a non-identity element $g \in G$ such that $Sg = S$. Since S is G_α-regular, we have that $g \in G - G_\alpha$. Therefore, by Claim 5, we conclude that $|S|$ is a multiple of p.

Alternatively, let $S' = S \cup \{\alpha\}$. Then we have that $S'_\alpha = (S'_\alpha)^{-1}$ and that S' is G_α-regular. Hence, similarly as in the case of S, we conclude that $|S'|$ is a multiple of p. Thus we obtain a contradiction because of $|S'| = |S| + 1$. This completes the proof of Gluck's theorem.

ACKNOWLEDGMENTS

This work was done during my stay at the University of Chicago. I express my hearty thanks to Prof. G. Glauberman for his kind hospitality and useful advice. I also appreciate the hospitality of Prof. J. L. Alperin and the members of the Department of Mathematics of the University of Chicago.

REFERENCES

1. H. Bender and G. Glauberman, "Local Analysis for the Odd Order Theorem," Cambridge Univ. Press, Cambridge, UK, 1994.
2. W. Feit and J. G. Thompson, Solvability of groups of odd order, *Pacific J. Math.* **13** (1963), 775–1029.
3. D. Gluck, Trivial set-stabilizers in finite permutation groups, *Canad. J. Math.* **35** (1983), 59–67.
4. T. R. Wolf, Solvable and nilpotent subgroups of $GL(n, q^m)$, *Canad. J. Math.* **34** (1982), 1097–1111.